Research Statement of Emanuel Milman

My main research interests lie in the various aspects of Asymptotic Convex Geometric Analysis. This is the study of geometric structures satisfying appropriate convexity conditions from a geometric and analytic point of view, with an emphasis on the asymptotic dependence (or independence) of various parameters on the underlying dimension. Examples of such structures include bounded convex domains (bodies) in Euclidean space $\mathbb{R}^n$, Banach spaces (possibly infinite dimensional), Riemannian manifolds with non-negative (Ricci) curvature, and other generalizations. Since its conception at the intersection of classical Convex Geometry and the local theory of Banach spaces, the field of Asymptotic Convex Geometric Analysis has been evolving constantly, and has uncovered connections to many other fields, such as Probability Theory, PDE, Riemannian Geometry, Harmonic Analysis, Mathematical Physics, Combinatorics, Graph Theory and Learning Theory. A general theme central to many questions in this field is that of "measure concentration": show that convexity forces most of the volume distribution of the geometric structure to concentrate in some canonical way (various ways to measure this concentration will be described later on); it is believed that, under certain natural normalizations, the answer to many fundamental questions should be *independent* of dimension.

One of my main research directions can be very roughly described as studying the interplay between a probability measure μ and a metric d defined on a common measure-metric space (Ω, d, μ). There are various ways to measure this relationship, such as via isoperimetric inequalities (comparing between the boundary-measure and the measure of sets), functional inequalities (such as those of Poincaré, Sobolev–Gagliardo–Nirenberg or log-Sobolev type), and concentration inequalities (studying the rate of tail-decay $\mu\left\{|f - \int f d\mu| > t\right\}$ for 1-Lipschitz functions f). It is known that isoperimetric inequalities imply their functional counterparts, which in turn imply appropriate concentration inequalities, and that these implications cannot be reversed in general. I was able to show that when (Ω, d, μ) is a Riemannian manifold equipped with a measure, which together satisfy some appropriate convexity assumptions, these implications can be reversed in a very strong sense: any *arbitrarily weak* concentration assumption implies a (linear) isoperimetric inequality, with dimension *independent* bounds. Previous results could only deduce such statements with (rather bad) dimension dependence, and this is the first (and to this date, only) result in this spirit which agrees with the Asymptotic Convex Geometric Analysis philosophy. To demonstrate the usefulness of this result, I obtained a sharp stability result for the spectral-gap of the Neumann Laplacian on convex domains in Euclidean space, and easily recovered and extended many known lower bounds on the spectral-gap, in a single unified framework.

I believe that my main Mathematical contribution lies in the successful application of results and techniques from the realm of Riemannian Geometry and Geometric Measure Theory (following ideas going back to M. Gromov), to tackle these types of problems, which have been mostly studied by the probability-oriented community, and that the full strength of this approach has yet to be revealed.

I have also been involved in the study of several fundamental questions on the asymptotic distribution of volume inside convex bodies in $\mathbb{R}^n$, when n tends to infinity. For instance, is it true that any convex body K of volume 1 in $\mathbb{R}^n$ has an $(n-1)$-dimensional section having volume uniformly bounded from below, independently of K or n? Is there some Central-Limit law for the one-dimensional projection (marginal) of the uniform distribution inside a convex body onto some direction? Can one compare the volumes of two (centrally symmetric) convex bodies by only comparing the volumes of their k-dimensional central sections $(1 < k < n)$? Can one efficiently cover a (centrally symmetric) convex body K by few translates of another convex body L, knowing that the polar (dual) body $L^\circ = \{x \in \mathbb{R}^n; \langle x, y\rangle \leq 1 \; \forall y \in L\}$ can be efficiently covered by K°? Although great progress has been recently achieved in some of these questions, many of them are still open in full generality, and I was able to obtain the best known quantitative results for several natural classes of convex bodies. Despite their geometric formulation, a broad array of methods from other fields was required for progressing in these directions, such as Analysis, Banach Space Theory, Probability Theory and even Integral Geometry.

Some of my *other* research interests include classical Convex Geometry, the interplay between geometry and spectral properties of Riemannian manifolds, the geometry of isoperimetric minimizing surfaces, diffusion semi-group and heat-kernel estimates, optimal transportation for the Monge-Ampère equation, empirical processes and metric entropy, asymptotic theory of finite-dimensional normed spaces ("Local Theory"), general phenomena in high dimensions, Sobolev spaces and Harmonic Analysis on Grassmann manifolds.

1 Isoperimetric, Functional and Concentration Inequalities

Isoperimetric, functional and concentration inequalities constitute different ways of measuring the interplay between a metric d defined on a space Ω and a Borel measure μ on (Ω, d). Isoperimetric inequalities compare between the boundary measure $\mu^+(A)$ of a Borel set A with its measure $\mu(A)$; for instance, the Euclidean isoperimetric inequality asserts that the Euclidean ball has minimal surface-area among all sets with given volume. Functional inequalities compare between the N_1 norm of a function and the N_2 norm of its gradient, where N_i are some norms defined on a space of nice functions, such as $L_p(\mu)$ norms; well-known examples include Poincaré, Sobolev–Gagliardo–Nirenberg and log-Sobolev inequalities. Concentration inequalities measure how tightly Lipschitz functions f are concentrated about their mean value, and provide quantitative estimates on the rate of tail decay $\mu\{|f - \int f d\mu| > t \|f\|_{Lip}\}$ (here $\|f\|_{Lip} := \sup_{x,y\in\Omega} \{|f(x) - f(y)| / d(x, y)\}$).

It is known from results of Maz'ya, Cheeger, Gromov–Milman, Gross, Herbst, Ledoux, Beckner, Bobkov and others, that in a very general setting, many known isoperimetric inequalities imply their functional counterparts, which in turn imply corresponding concentration inequalities, and that in general these implications cannot be reversed. One of my main research objectives is to show that when (Ω, d, μ) satisfies some appropriate convexity assumptions, these implications can in fact be reversed with dimension *independent* bounds.

The simplest manifestation of convexity is when Ω is a convex bounded domain in $\mathbb{R}^n$, the metric d is given by a Euclidean structure, and μ is the Lebesgue measure restricted to Ω and normalized to have total measure 1. In that case the boundary measure $\mu^+(A)$ coincides (up to scaling) with the usual interior surface-area given by the Hausdorff measure $H_{n-1}(\partial A \cap \Omega)$, and this is the case which the reader should keep in mind. More generally, the results below also apply to measures μ with log-concave densities, uniform measures on locally convex domains in Riemannian manifolds with non-negative Ricci curvature (with d the induced geodesic distance), and the more general Bakry-Émery curvature dimension conditions, which combine information from both geometry and measure.

1.1 Uniform tail decay implies a linear isoperimetry inequality

In [2], as announced in [1], I showed that under these convexity assumptions, the following four properties are equivalent (in the sense that $D_{Che} \leq c_1 D_{Poin} \leq c_2 D_{Exp} \leq c_3 D_{Lin} \leq c_4 D_{Che}$, with D_* being the best constants in the statements below, and $c_i > 0$ universal constants, *independent* of the dimension n):

1. Linear isoperimetric inequality: $\mu^+(A) \geq D_{Che} \min(\mu(A), 1-\mu(A)) \quad \forall A \subset \Omega$.
2. Poincaré inequality: $\int |\nabla f|^2 d\mu \geq D_{Poin}^2 \int |f - \int f|^2 d\mu \quad \forall$ nice f.
3. Exponential concentration: $\mu(|f - \int f| > t \left\|f\right\|_{Lip}) \leq 2\exp(-D_{Exp} t) \;\; \forall f \; \forall t > 0$.
4. Linear concentration: $\mu(|f - \int f| > t \left\|f\right\|_{Lip}) \leq \frac{1}{D_{Lin} t} \;\; \forall f \; \forall t > 0$.

The implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) are classical and follow respectively in a general setting from the work of Maz'ya and Cheeger (independently), and Gromov–Milman. (3) $\Rightarrow$ (4) is trivial, and the other implications are false in general. Under the additional assumption of convexity, the implication (2) $\Rightarrow$ (1) was previously shown by Buser and extended by Ledoux. I was able to show that in fact, a much weaker assumption (4) is required to obtain the same conclusion (1): any *arbitrarily slow* uniform rate of tail-decay of Lipschitz functions suffices (implying in particular that this rate must have actually been exponentially fast (3)); the linear tail-decay used in (4) is just for concreteness. To this date, this is the only instance of a result showing that concentration implies isoperimetry with dimension *independent* bounds.

To demonstrate the usefulness of this result outside the study of isoperimetric inequalities, I proved as an application a quantitatively sharp stability result for the spectral-gap $\lambda_1^N(\Omega)$ of the Laplacian $-\Delta$ on Ω with Neumann boundary conditions (it is classical that $\lambda_1^N(\Omega) = D_{Poin}^2(\Omega)$ in the notation of (2)). In general, there can be no stability for $\lambda_1^N(\Omega)$ under perturbations of the domain Ω, but for convex domains $K_1 \subset K_2$ in Euclidean space, I was able to show:

$$\mathrm{Vol}(K_1) \geq v \mathrm{Vol}(K_2) \;\Rightarrow\; cv^2 \leq \sqrt{\frac{\lambda_1^N(K_2)}{\lambda_1^N(K_1)}} \leq C \log\left(1 + \frac{1}{v}\right) \;,$$

where $c, C > 0$ are universal constant and Vol denotes volume. As further applications, I also obtained simple proofs recovering and extending many lower bounds on λ_1^N due to Payne–Weinberger, Li–Yau, Kannan–Lovász–Simonovits, Bobkov and S. Sodin, in a single unified framework.

The methods involved in the proofs include diffusion semi-group estimates following Ledoux and a novel application of some recent results in Riemannian Geometry.

1.2 Isoperimetric and functional inequalities are equivalent

In [3], I treated more general isoperimetric and functional inequalities (involving arbitrary Orlicz norms), and showed that under the convexity assumptions, they are equivalent one to the other with dimension *independent* bounds. Treating this more general case was technically more involved, and required developing a new coherent framework for passing between isoperimetric, functional and *isocapacitary* inequalities, which were introduced by Maz'ya, and used by Barthe–Cattiaux–Roberto. As usual, that isoperimetric inequalities imply their functional counterparts does not require any convexity assumptions and holds in a general setting, but even this direction proved to be technically challenging.

As a concrete application, I showed that under our convexity assumptions, q-log-Sobolev inequalities (which generalize the usual log-Sobolev inequality of Gross from $q = 2$ to $q \in [1, 2]$) are equivalent to an appropriate family of isoperimetric inequalities (which generalize the Gaussian one), extending results of Bakry–Ledoux and Bobkov–Zegarlinski. As a further application, I addressed the question of characterizing when an isoperimetric inequality on (Ω, d, μ) remains also valid (up to some dimension *independent* bounds) on the ℓ_2-product space $(\Omega^n, d^{\otimes n}, \mu^{\otimes n})$. This is closely related to the Central-Limit theorem, since one cannot obtain an isoperimetric inequality which is better than the Gaussian one as n tends to infinity. I extended the known results on this question due to Barthe–Cattiaux–Roberto, giving a new sufficient condition corresponding to the absence of a Central-Limit obstruction.

1.3 Projects in progress

Following works of Gromov and Buser, I have been recently able to prove [15] the above mentioned results without appealing to the Bakry–Ledoux diffusion semi-group method, relying solely on Geometric Measure Theory and Riemannian Geometry. Moreover, the geometric method allows to extend to the Riemannian setting some results of Bobkov and Kannan–Lovász–Simonovits, which were previously confined to the Euclidean one, and lends itself to further generalizations. For instance, this method permits imposing Dirichlet boundary conditions, in which case the appropriate "convexity" property required for reversing the general implications turns out to be the requirement that the boundary have non-negative mean-curvature (which is weaker than the convexity required in the absence of boundary conditions).

In a joint work with Bo'az Klartag [16], we used this property to study the spectral-gap $\lambda_1^D(\Omega)$ of the Laplacian $-\Delta$ with Dirichlet boundary conditions on Ω. The well known Faber–Krahn inequality asserts that $\lambda_1^D(\Omega) \geq \lambda_1^D(\Omega^*)$ where Ω^* denotes a Euclidean ball having the same volume as Ω. For a convex domain Ω, by allowing to change the Euclidean structure if necessary, we obtained a reverse form of the Faber–Krahn inequality, which is (almost) dimension independent. This also implies a reverse form of an inequality due to Brascamp–Lieb, which is an analogue of the Brunn-Minkowski inequality for λ_1^D.

1.4 Isoperimetric inequalities on uniformly convex bodies

In a joint work with Sasha Sodin [4], we proved an (essentially sharp) isoperimetric inequality on a uniformly convex body K, depending only on its modulus of convexity, where the measure μ is uniform on K and the metric is given by $d(x,y) = \|x-y\|_K$ (the norm whose unit-ball is K, as opposed to the Euclidean metric used above). A convex body is called uniformly convex if its boundary is strictly convex (a tangent plane has a single point of contact), and its associated modulus of convexity measures this quantitatively. This filled the final missing piece in the isoperimetry-functional inequality-concentration picture for this scenario, by generalizing an earlier concentration result due to Gromov–Milman and a functional log-Sobolev type inequality of Bobkov–Ledoux, and improving a non-sharp isoperimetric inequality of Bobkov–Zegarlinski. The ideas involved naturally led to a novel notion of *uniformly* log-concave measures, and extended the known analogy between convex bodies and log-concave densities. Our isoperimetric inequality in fact also applies more generally to the class of uniformly log-concave measures, generalizing an isoperimetric inequality of Bakry–Ledoux. Our method relied on a geometric localization argument of Kannan–Lovász–Simonovits (which in fact can be traced in more general form to the work of Gromov–Milman).

2 The Slicing Problem

One of the most well-known and intriguing problems in Asymptotic Convex Geometry is the Slicing Problem, originally posed by J. Bourgain in the 1980's and studied by Milman–Pajor. It asks whether any convex body K in $\mathbb{R}^n$ of volume 1, has an $(n-1)$-dimensional section whose volume is bounded from below by a universal constant $c > 0$ not depending on K or n. This problem is known to have many equivalent formulations. For instance, it is equivalent to the existence of an ellipsoid $\mathcal{E}$ with $\mathrm{Vol}(K) = \mathrm{Vol}(\mathcal{E})$ such that $\mathrm{Vol}(K \cap C\mathcal{E}) \geq \mathrm{Vol}(K)/2$ (any constant fraction smaller than 1 will also do), where $C < \infty$ is a universal constant. The problem is known to have a positive answer for many families of convex bodies, but the general case is still open; the best known quantitative bound on c (or C) as a function of n is due to Bo'az Klartag, slightly improving a bound of Bourgain.

In [11], I developed a technique using dual mixed-volumes (which were introduced by E. Lutwak in the 1970's) for attacking this problem, by comparing K with a (non necessarily convex) body L containing K, which is chosen from a less general family of bodies. In particular, this approach recovers, strengthens and generalizes results of K. Ball and M. Junge for unit-balls of subspaces and quotients of L_p for $1 < p < \infty$ (in a subsequent joint work with Klartag [7], we managed to further improve Junge's bounds for quotients of subspaces of L_p for $1 < p < 2$). I also extended these results to negative values of p using generalized intersection-bodies, described in Section 4.

3 Duality of Entropy

By an easy combination of previously known results, I was able to strengthen in [8] the best known (to this date) quantitative bounds in the Duality of Entropy conjecture. Originally posed by Pietsch in the 1970's in the language of entropy numbers of compact operators on Banach spaces, this conjecture can be equivalently formulated geometrically using the notion of covering numbers. Given two centrally symmetric convex bodies K and T, the covering number of K by T, denoted $N(K,T)$, is defined as the minimal number of translates of

T needed to cover K. The conjecture then states that there exist two universal constants $a, b > 1$, which do not depend on any parameter, so that $\log N(K,T) \le b \log N(T^\circ, K^\circ/a)$. Here L° denotes the polar (or dual) body to L, defined as $L^\circ = \{x \in \mathbb{R}^n; \langle x, y\rangle \le 1 \; \forall y \in L\}$. There has been much progress in the study of this conjecture in recent years due to the work of Artstein, Milman, Szarek and Tomczak-Jaegermann, utilizing ideas from previous work of Bourgain–Pajor–Szarek–Tomczak-Jaegermann, and it was shown to hold for a rich family of convex bodies (including the important Euclidean ball). Nevertheless, for general convex bodies in $\mathbb{R}^n$, the best known bounds for a, b remained polynomial in n. Employing some forgotten results in these previous works, I noted that this could be substantially improved to a logarithmic dependence. I also showed how to extend this duality to Talagrand's γ_p functionals ($p > 0$), which appear naturally in the study of stochastic and empirical processes.

4 Low-Dimensional Busemann-Petty Problem

Another (partially) open question about volumes of sections is a generalization of the Busemann-Petty problem posed by R. Gardner and G. Zhang in the 1990's. It asks the following: if K and L are two centrally symmetric convex bodies in $\mathbb{R}^n$, such that $\mathrm{Vol}(K \cap E) \le \mathrm{Vol}(L \cap E)$ for all k dimensional linear subspaces E, does it follow that $\mathrm{Vol}(K) \le \mathrm{Vol}(L)$? Here Vol denotes volume in the corresponding affine hull. The original problem of Busemann-Petty (dating back to 1956) asked about the case $k = n - 1$, and was answered in a series of contributions by many authors (including Larman, Rogers, Ball, Bourgain, Giannopoulos, Papadimitrakis, Lutwak, Gardner, Zhang, Koldobsky and Schlumprecht) as follows: the answer is positive if $n \le 4$ but surprisingly negative for $n \ge 5$. The generalized problem was shown to have a negative answer when $3 < k < n$ by Bourgain and Zhang, but the cases $k = 2, 3$ are still open. When L is a Euclidean ball and K is a small perturbation of it, Bourgain–Zhang showed that the answer is positive in the case $k = 2$.

In [9], I gave a complementary result by showing that the answer is positive in the cases $k = 2, 3$ when K and L are (non-necessarily convex) star-bodies, such that K's radius in each direction is the $(n-k)$-th root of the radial function of a convex body. In particular, this implies that when K is sufficiently close to a Euclidean ball (to an extent depending on its curvature), the answer is positive for any star-body L.

It is also interesting to note that a positive answer to the generalized BP problem for $k = 2, 3$ would follow from an equivalence between two generalizations of intersection-bodies. Intersection-bodies were introduced by E. Lutwak and played a key role in the solution of the original BP problem (unfortunately, I will not define them here). The natural geometric generalization of this problem to arbitrary k, led Zhang to introduce one type of generalized intersection-bodies. A second type was introduced by A. Koldobsky, who studied a different analytic generalization of this problem. Koldobsky also studied the connection between these two types of generalizations, and noted that an equivalence between these two notions would completely settle the unresolved cases $k = 2, 3$ in the generalized BP problem.

In [10], I showed that these classes share many identical structure properties, proving the same results using Integral Geometry techniques for Zhang's class and Fourier transform of distributions techniques for Koldobsky's class. Using a Functional Analytic approach, I gave several surprising equivalent formulations for the equivalence problem, which revealed an intimate connection to several fundamental problems on the Integral Geometry of the Grassmann manifold.

Although these results provided substantial evidence towards a positive answer to the equivalence question, I constructed a surprising counter-example in [5], which settled the question in the negative. This result implies in particular the existence of non-trivial non-negative functions in the range of the spherical Radon transform, and the existence of non-trivial spaces which embed in L_p for certain negative values of p.

5 Central Limit Problem for Convex Bodies

The Central-Limit Problem for convex bodies, whose origins may be traced back to ideas of Gromov, was explicitly stated by Brehm–Voigt and Antilla–Ball–Perissinaki. It asks whether for every convex body K in $\mathbb{R}^n$ there exists a direction, such that the one-dimensional projection (marginal) of the uniform distribution in K onto that direction has approximately Gaussian distribution, with an increasing level of approximation as n tends to infinity. Of course this cannot hold for any direction as witnessed by the n-dimensional cube, whose marginals in the direction of the axes are always uniform distributions. This problem is a generalization of the classical Central-Limit law to the case where the n random-variables are not independent, but rather represent an n-dimensional vector uniformly distributed inside a convex body.

In a joint work with Bo'az Klartag [7], we improved previous results of Antilla–Ball–Perissinaki, and provided a positive answer for the class of uniformly convex bodies having quadratic modulus of convexity. The latter constitutes a rich (affine invariant) family, and includes the unit balls of l_p^n for $1 < p \leq 2$. In a subsequent work [6], I strengthened and extended this result to more general convex bodies. I showed the existence of Gaussian marginals in a strong sense for uniformly convex bodies having modulus of convexity of type p with $2 \leq p < 4$, by using a log-Sobolev type inequality of Bobkov–Ledoux and an application of Talagrand's "Majorizing Measures Theorem". Under some additional assumptions, I also extended this to values of p greater than 4, showing in particular that all unit-balls of subspaces of quotients of L_p for $1 < p < \infty$ have Gaussian marginals.

After this work was completed, Klartag obtained a remarkable positive solution to the Central-Limit problem for arbitrary convex bodies. Nevertheless, his general quantitative estimates remain inferior to those obtained in most of the results mentioned above, and the question of what are the optimal estimates remains open.

6 Game Theory

In another direction, my M.Sc. Thesis, entitled "Uniform Properties of Stochastic Games and Approachability", concerned two-player zero-sum finite-state stochastic games. In [13], I used the theory of semi-algebraic sets and mappings to prove some asymptotic properties of the min-max value, which hold uniformly for all stochastic games given an upper bound on the number of states and players' actions. As a corollary, I proved a uniform polynomial convergence rate of the value of the N-stage game to the value of the non-discount game, over a bounded set of payoffs, thereby extending a celebrated result of Mertens–Neyman. In [12], I used the latter result in the study of approachable sets in vector-payoff games, generalizing a sufficient condition for approachability of Blackwell to the wider context of multiple state games. In the case of a convex set, I showed that the sufficient condition for approachability is also necessary.

7 Future Plans

There are several exciting directions for continuing my current line of research, especially in view of recent advances in the study of several fundamental problems in the field.

7.1 Concentration implies isoperimetry

My results from Subsection 1.1 show that under convexity assumptions, a uniform tail-decay of 1-Lipschitz functions implies a linear isoperimetric inequality. This is the first (and to this date, the only) instance of such an implication where the resulting bounds are dimension *independent*. It is known from the works of Bobkov, Ledoux, Wang, Barthe, Kolesnikov and others that under additional fast tail-decay assumptions (e.g. sub-Gaussian), a stronger isoperimetric inequality (e.g. Gaussian isoperimetric inequality) may be deduced, but all of the known approaches produce (rather bad) dimension dependent bounds. It would be very interesting to extend our geometric approach to attack this problem, and obtain dimension *independent* bounds. This would also have implications concerning transportation-cost inequalities, due to results of Marton, Talagrand, Bobkov–Götze, Otto–Villani, Gozlan and others. In a very recent progress [14], I succeeded to prove this result using a geometric volume comparison theorem.

7.2 Extending results to semi-convex and graph settings

It would also be interesting to extend the results of Section 1 to more general assumptions than convexity. For instance, one could only assume a (negative) lower bound on the Ricci curvature, or more generally consider the Bakry–Émery curvature-dimension condition $CD(R, N)$, with negative values for R or N (the latter possibility has been essentially introduced by Borell and Brascamp–Lieb, and developed by Bobkov and Bobkov–Ledoux). Another possibility would be to consider a non Riemannian setting, by using the recent generalizations of Ricci curvature to the measure-metric space setting introduced by Sturm, Lott–Villani, Ollivier and others. An especially interesting case is the graph-setting, where one could hope to obtain new results connecting spectral-gap to other natural parameters.

7.3 Variance Conjecture

The Central Limit Problem described in Section 5 is closely related to the Variance Conjecture, which asks whether most of the volume inside an isotropic convex body K of volume 1 lies in an annulus of radii $(1-\varepsilon)\rho$ and $(1+\varepsilon)\rho$, with ε tending to 0 as the dimension n tends to infinity. A body is called isotropic if the variance of its one-dimensional marginals (i.e.

projection of the uniform distribution in the body onto a one-dimensional subspace) does not depend on the marginal's direction; it is known that every body has such an affine image. In his solution to the CLP, Klartag obtained an estimate of the form $\varepsilon = O(1/n^{1/10-\delta})$, but the Variance Conjecture predicts that this rate should be of the order $\varepsilon = O(1/n^{1/2})$. In a joint effort with Sasha Sodin, still in progress, we have developed an approach following ideas of Klartag and Paouris which might be able to improve the best known bounds.

7.4 Kannan-Lovász-Simonovits Conjecture

Motivated by algorithmic questions, Kannan, Lovász and Simonovits studied the mixing time of a certain random walk inside a convex body K in $\mathbb{R}^n$, which is essentially governed by the Cheeger constant (defined in Section 1) of K with respect to the Euclidean metric, denoted $D_{Che}(K)$. KLS conjectured that one always has $D_{Che}(K) \geq c/L_K$ for an isotropic convex body K of volume 1, and managed to prove that $D_{Che}(K) \geq c/(\sqrt{n}L_K)$, for some universal constant $c > 0$. By combining Klartag's variance bound with a recent result of Bobkov, it is possible to improve this bound to $D_{Che}(K) \geq c_\delta/(n^{1/2-1/20+\delta}L_K)$, and our results for uniformly convex bodies give even better results for the latter class of bodies. A positive answer to the Variance Conjecture would lead to the bound $D_{Che}(K) \geq c/(n^{1/4}L_K)$, and perhaps with our recent results from Section 1, it would be possible to approach this bound. Indeed, for isotropic convex bodies which are symmetric with respect to reflection about the coordinate hyperplanes, it was shown by Klartag that our stability result from Subsection 1.1 can be used to obtain the bound $D_{Che}(K) \geq c/\log(1+n)$. It would be interesting to try to settle the conjecture for this class, or at least for the subclass of bodies which are in addition invariant under coordinate permutations.

7.5 Construct a counter-example to the Slicing Problem

The Slicing Problem, described in Section 2, is a fundamental question on the asymptotic distribution of volume inside a convex body. The big gap between the known and conjectured estimates for this problem is very intriguing, and it is essential to progress by closing the gap, in either the positive or negative directions. It seems that the negative direction has not been explored as extensively as the positive one, and offers an excellent research opportunity.

7.6 Low-dimensional Busemann-Petty problem

The unresolved cases $k = 2, 3$ in the generalized Busemann-Petty problem, described in Section 4, also offer a good research direction. As indicated by my partial positive answer to these cases, it seems that a fusion between the analytical methods developed by Koldobsky and Integral Geometric methods is needed in order to obtain further progress on this problem. In addition, the negative answer I obtained to the equivalence problem of generalized intersection-bodies, suggests that one should also keep an open mind about the possibility that there exists a counter-example to this problem as well.

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