054414 Process Control System Design

LECTURE 9: ANALYSIS OF DISCRETE SYSTEM RESPONSE

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Objectives

On completing this section, you should:

❶ Be able to represent continuous and discrete signals in the same framework (Z-transforms).

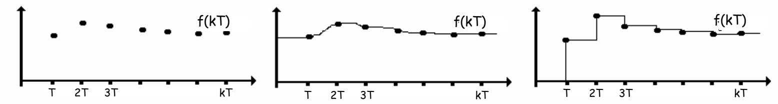
❷ Have a working knowledge of Z-transforms and their usage in the analysis of discrete systems.

❸ Be able to compute the transfer function of an arbitrary system involving continuous and discrete components.

❹ Be able to compute the response of a discrete system (open and closed loop).

Signal Representation

Must differentiate between:
❶ Purely discrete signals.
❷ Sampled continuous signals.
❸ Continuous representations of discrete signals (hold).

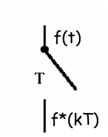


(1) Discrete (2) Sampled (3) Hold

All 3 representations are needed to model system components.

Quiz: Which representation best describes the following:
❶ The solution of an ODE obtained by numerical integration.
❷ The output of a D/A unit.
❸ The composition measurement obtained from a GC.
❹ The output of a A/D unit.

Sampling and Signal Reconstruction



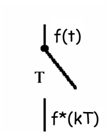
After sampling, the continuous time signal, f(t), is a sequence of impulses, f*(t), with the magnitude of each impulse being the magnitude of the original signal at the sample instant.

Note that the action of the sampler is such that at $t = t_k$, the sampler returns a value $f^*(t_k)$, for the small time interval (t_ε) that it is closed, and zero elsewhere.

Assuming an ideal sampler operation ($t_\varepsilon = \varepsilon \rightarrow 0$), then a simple sampler output at time t_k is most conveniently modeled using the ideal delta function:

$$f^*(t_k) = \int_{t_k-\varepsilon}^{t_k+\varepsilon} f(t)\delta(t-t_k)\,dt \quad \text{since} \quad \int_{t-\varepsilon}^{t+\varepsilon} x(t)\delta(t-\theta)\,dt = \begin{cases} x(\theta), & t=\theta \\ 0, & t\neq\theta \end{cases}$$

1 Sampling and Signal Reconstruction



A single "spike" in the sampled sequence is:

$$f^*(t_k) = \int_{t_k-\varepsilon}^{t_k+\varepsilon} f(t)\delta(t-t_k)\,dt$$

What is observed is a sequence of impulses, do that the complete sampled signal is the series:

$$f^*(t) = \int_0^{\infty} \sum_{k=0}^{\infty} f(t)\,\delta(t-t_k)\,dt = f^*(t_0) + f^*(t_1) + \ldots + f^*(t_k) + \ldots$$

The above can be written more compactly as follows:

$$f^*(t) = \sum_{k=0}^{\infty} f(t_k)\,\delta(t-t_k)$$

Laplace transform gives: $L\{f^*(t)\} = F^*(s) = \sum_{k=0}^{\infty} f(t_k) L\{\delta(t-t_k)\}$

$$= \sum_{k=0}^{\infty} f(t_k) e^{-kTs}$$

1 Sampling and Signal Reconstruction

Laplace transform gives: $L\{f^*(t)\} = F^*(s) = \sum_{k=0}^{\infty} f(t_k) e^{-kTs}$

$$= \sum_{k=0}^{\infty} f(t_k) z^{-k}$$

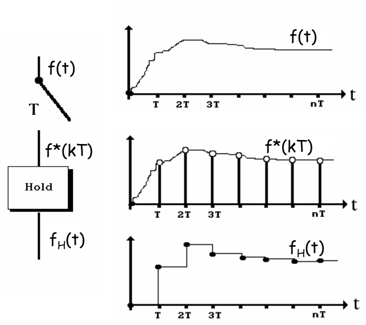
Definition of Z-transform (see part ❷ of this lecture)

Note the relationship: $z = e^{Ts}$

This will be using this later to assist in the analysis of stability and performance of discrete systems.

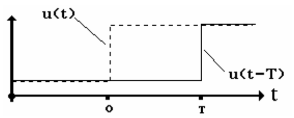
1 Representation of Zero-order Hold

A device that converts impulses f(t) at times t = kT into pulses of height f*(kT) and width T.



By definition:

$$h(t) = u(t) - u(t-T)$$



Taking Laplace transform:

$$H(s) = L\{h(t)\} = L\{u(t) - u(t-T)\} = \frac{1-e^{-Ts}}{s}$$

2 Z-Transforms

Z-transforms play the same role for discrete systems as that played by Laplace transforms for continuous systems.

Definition: $Z\{y(kT), k = 0,1,2,...\} = \sum_{k=0}^{\infty} y(kT) z^{-k}$

The "right-sided" sequence, which assumes y(kT) = 0, k < 0.

Notes:
❶ z is a complex variable.
❷ Z{ } transforms the function y(kT) of a discrete variable, kT, into a continuous function of a complex variable, z.
❸ If two different signals, $y_1(t)$ and $y_2(t)$, have the same sampled values y(kT), they also have the same transform.
❹ Z{ } is only valid if $\sum_{k=0}^{\infty} y(kT) z^{-k}$ converges.
❺ $z = e^{Ts}$

2 Common Z-Transforms

Unit Step. $y(kT) = 1$

$$Y(z) = 1 + z^{-1} + z^{-2} + \ldots = \sum_{k=0}^{\infty} z^{-k}$$

$$= \frac{1}{1 - z^{-1}} = \frac{z}{z-1} \text{ provided } |z^{-1}| < 1$$

Exponential. $y(kT) = e^{-akT} = b^{-k}, b = e^{aT}$

$$Y(z) = Z\{e^{-akT}\} = \sum_{k=0}^{\infty} b^{-k} z^{-k} = \sum_{k=0}^{\infty} (bz)^{-k}$$

$$= \frac{1}{1 - (bz)^{-1}} = \frac{z}{z - e^{-aT}} \text{ provided } |(bz)^{-1}| < 1$$

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2 Common Z-Transforms

Ramp. $y(kT) = akT = ck, c = aT$

$$Y(z) = Z\{akT\} = \sum_{k=0}^{\infty} akTz^{-k} = aT\sum_{k=0}^{\infty} kz^{-k}$$

$$= aT\sum_{k=0}^{\infty} kz^{-(k-1)} z^{-1} = aT\left(1 + 2z^{-1} + 3z^{-2} + \ldots\right) z^{-1}$$

$$= \frac{aTz^{-1}}{\left(1 - z^{-1}\right)^2} = \frac{aTz}{(z-1)^2} \text{ provided } |z^{-1}| < 1$$

Cosine. $y(kT) = \cos(\omega kT) = \tfrac{1}{2}\left(e^{i\omega kT} + e^{-i\omega kT}\right), k > 0$

$$Y(z) = Z\{\cos(\omega kT)\} = \frac{1}{2}\left(\frac{1}{1 - e^{i\omega T} z^{-1}} + \frac{1}{1 - e^{-i\omega T} z^{-1}}\right)$$

$$= \frac{1 - \cos\omega Tz^{-1}}{1 - 2\cos\omega Tz^{-1} + z^{-2}} = \frac{z^2 - z\cos\omega T}{z^2 - 2z\cos\omega T + 1}$$

Sine. $Z\{\sin(\omega kT)\} = \dfrac{\sin\omega Tz^{-1}}{1 - 2\cos\omega Tz^{-1} + z^{-2}} = \dfrac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1}$

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2 Properties of Z-Transforms

Linearity. $Z\{y_1(kT)+a\cdot y_2(kT)\} = Z\{y_1(kT)\}+a\cdot Z\{y_2(kT)\}$

Delay. $$Z\{y((k-n)T)\} = \sum_{k=0}^{\infty} y(kT)z^{-(k+n)}$$
$$= z^{-n}\sum_{k=0}^{\infty} y(kT)z^{-k} = z^{-n}Z\{y(kT)\}$$

Advance. $Z\{y(k+1)\} = zY(z)-zy(0)$

Proof: $$Z\{y(k+1)\} = \sum_{k=0}^{\infty} y(k+1)z^{-k} = z\sum_{k=0}^{\infty} y(k+1)z^{-(k+1)}$$
$$= z\Big[\underbrace{y(0)+\sum_{k=0}^{\infty} y(k+1)z^{-(k+1)}}_{Y(z)} - y(0)\Big]$$

In general:
$$Z\{y(k+n)\} = z^nY(z)-z^ny(0)-z^{n-1}y(1)-\ldots-zy(n-1)$$

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2 Final Value Theorem (FVT)

$\lim_{k\to\infty} y(k) = \lim_{z\to 1}(1-z^{-1})Y(z)$ Only if Y(z) is stable

Example 9.1

$$G(z) = \frac{Y(z)}{U(z)} = \frac{0.25z}{(z-0.5)^2} = \frac{z}{(2z-1)^2}$$

Suppose $U(z) = \dfrac{2z}{z-1}$ (A step of magnitude 2)

$$Y(z) = G(z)\cdot U(z) = \frac{2z^2}{(2z-1)^2(z-1)}$$

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1}(1-z^{-1})Y(z) = \lim_{z\to 1}\frac{z-1}{z}\,\frac{2z^2}{(2z-1)^2(z-1)}$$
$$= 2$$

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2 Initial Value Theorem (IVT)

If $y(k)=0$ for $k<0$, then $y(0)=\lim_{z\to\infty} Y(z)$

Example 9.2

$$Y(z)=G(z)\cdot U(z)=\frac{2z}{(z-1)}\frac{z}{(2z-1)^2}=\frac{2}{(1-z^{-1})}\frac{0.25z^{-1}}{(1-0.5z^{-1})^2}$$

Hint: Need to express the transfer function in terms of z^{-1}

$$\text{Hence, } y(0)=\lim_{z\to\infty} Y(z)=\lim_{z\to\infty}\frac{2}{(1-z^{-1})}\frac{0.25z^{-1}}{(1-0.5z^{-1})^2}=0$$

2 Inversion of Z-Transforms

Inversion of z-transforms involves the generation of a discrete time-domain response from a rational z-transfer function...

Conceptually, given the rational form:

$$Y(z)=\frac{Q(z^{-1})}{R(z^{-1})}\begin{matrix}\leftarrow \text{polynomial of order m}\\ \leftarrow \text{polynomial of order n}\end{matrix}\Big\}\ n\ge m$$

... the numerator polynomial is divided by the denominator polynomial, to obtain:

$$Y(z)=\gamma_0+\gamma_1 z^{-1}+\gamma_2 z^{-2}+\dots$$

... which is of course, the discrete response itself! The alternatives to the generation of the discrete response are:

- Inversion by long division
- Inversion by Partial Fraction Expansion
- Simulation using SIMULINK

Inversion by Long Division

Conceptually, given the rational form, the numerator polynomial is divided by the denominator polynomial, to obtain the result directly.

Example 9.3 Using long division, compute the discrete response for:

$$Y(z) = \frac{z^{-1}}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

Solution.

$$\begin{array}{r|l} & z^{-1} + 1.5z^{-2} + 1.75z^{-3} + \ldots \\ \hline 1 - 1.5z^{-1} + 0.5z^{-2} & z^{-1} \\ & \underline{z^{-1} - 1.5z^{-2} + 0.50z^{-3}} \\ & + 1.5z^{-2} - 0.50z^{-3} \\ & \underline{+1.5z^{-2} - 2.25z^{-3} + 0.75z^{-4}} \\ & +1.75z^{-3} - 0.75z^{-4} \end{array}$$

Inversion by PFE

This involves a procedure similar to that used in Laplace transform inversion. Given:

$$Y(z) = \frac{Q(z^{-1})}{R(z^{-1})} \begin{array}{l} \leftarrow \text{polynomial of order m} \\ \leftarrow \text{polynomial of order n} \end{array} \Big\} \; n \geq m$$

The following steps are involved:

❶ Expansion:

$$Y(z) = \frac{Q(z^{-1})}{R(z^{-1})} = \frac{C_1}{r_1(z^{-1})} + \frac{C_2}{r_2(z^{-1})} + \ldots + \frac{C_n}{r_n(z^{-1})}$$

❷ Evaluation if C_1, C_2, …, C_n.

❸ Determine $y_i(k)$ for each individual term from a table of z-transforms (see slides 9-11, and tables in textbooks).

2 Inversion by PFE (Cont'd)

Example 9.4: First Order System.

$$a_1 y(i+1) + a_0 y(i) = u(i), y(0) = 0, u(i) = 1.$$

Taking z-transform: $(a_1 z + a_0) Y(z) = \frac{z}{z-1}$

Rearranging: $Y(z) = \frac{z}{(z-1)(a_1 z + a_0)} = \frac{(1/a_1) z^{-1}}{(1-z^{-1})(1-(-a_0/a_1) z^{-1})}$

❶ Expansion: $Y(z) = \frac{1}{a_1}\left[\frac{C_1}{1-z^{-1}} + \frac{C_2}{1-(-a_0/a_1) z^{-1}}\right]$

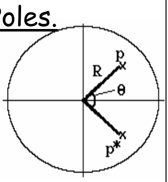
❷ $C_1 = \frac{a_1}{a_0 + a_1}, C_2 = -\frac{a_1}{a_0 + a_1}$.

❸ From slide 9: $y(k) = \frac{1}{a_0 + a_1}\left[1 - \left(-\frac{a_0}{a_1}\right)^k\right]$

2 Inversion by PFE (Cont'd)

Example 9.5: Second Order System with Complex Poles.

$$Y(z) = \frac{z^2 + z}{(z-0.8)(z^2 - 1.13z + 0.64)}$$



❶&❷ $Y(z) = \frac{A}{1-pz^{-1}} + \frac{A^*}{1-p^* z^{-1}} + \frac{B}{1-0.8z^{-1}}$

$p, p^* = 0.565 \pm 0.565i = 0.8e^{\pm i\frac{\pi}{4}}$ $A, A^* = -2.39 \pm 0.1147i$ $B = 4.78$

❸ $y(k) = (-2.39 + i0.115)\left(0.8e^{i\frac{\pi}{4}}\right)^k - (2.39 + i0.115)\left(0.8e^{-i\frac{\pi}{4}}\right)^k + 4.78(0.8^k)$

$$= 0.8^k(-2.39 + i0.115)\left(\cos\left(\tfrac{\pi}{4}k\right) + i\sin\left(\tfrac{\pi}{4}k\right)\right) - (2.39 + i0.115)\left(\cos\left(\tfrac{\pi}{4}k\right) - i\sin\left(\tfrac{\pi}{4}k\right)\right) + 4.78(0.8^k)$$

$$= 0.8^k\left(4.78 - \left(4.78\cos\left(\tfrac{\pi}{4}k\right) + 0.230\sin\left(\tfrac{\pi}{4}k\right)\right)\right)$$

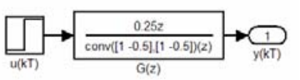
2 Inversion using MATLAB

Example 9.6:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{0.25z}{(z-0.5)^2}$$

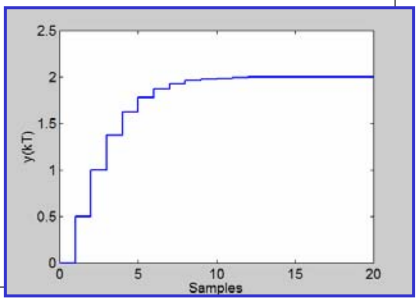
Compute the open-loop response of y(kT) to a step change of 2 units in u(kT). Verify that the final value of y(kT) approaches the value of 2.

Simulink model:



In MATLAB:

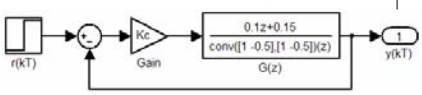
```
» ylabel('y(kT)')
» [t,x,y]=sim('Lec_8_ol',20);
» [tt,yy]=stairs(t,y);
» plot(tt,yy)
```



2 Inversion using MATLAB

Example 9.7: $G(z) = \frac{Y(z)}{U(z)} = \frac{0.1(z+1.5)}{(z-0.5)^2}$

Investigate the effect of controller gain on the CL response with proportional control. Check the results with the FVT.



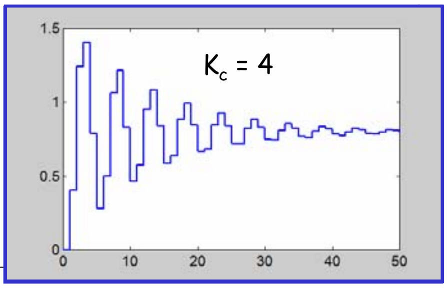
Solution:

$$\frac{Y(z)}{R(z)} = \frac{K_C G(z)}{1+K_C G(z)}$$

$$= \frac{0.1K_C(z+1.5)}{(z-0.5)^2 + 0.1K_C(z+1.5)}$$

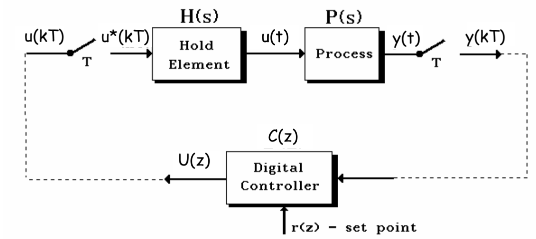
Since $R(z) = z/(z-1)$, the FVT gives:

$$\lim_{z\to1} Y(z) = \frac{K_C}{1+K_C}$$

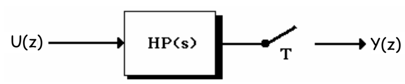


3 Block Diagrams/Transfer Functions

The most common set-up involves a sampled continuous process, controlled by a discrete controller, which sends its input via a zero-order hold (why?):

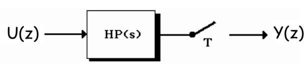


Because of the D/A conversion (mathematically described by the zero-order hold), the effective transfer function is:



3 Discrete Transfer Function

The transfer function between U(z) and Y(z) is:



$$G(z) = \frac{Y(z)}{U(z)} = Z\left\{L^{-1}\left\{H(s)P(s)\right\}\right\}$$

Using zero-order hold for H(s):

$$G(z) = \frac{Y(z)}{U(z)} = \left(1 - z^{-1}\right) Z\left\{L^{-1}\left\{\frac{P(s)}{s}\right\}\right\}$$

<u>Example:</u> $P(s) = \dfrac{K}{\tau s + 1}$

$$G(z) = K\left(1 - z^{-1}\right) Z\left\{L^{-1}\left\{\frac{1}{s(\tau s + 1)}\right\}\right\} = K\left(1 - z^{-1}\right) Z\left\{1 - e^{-t/\tau}\right\}$$

$$= K\left(1 - z^{-1}\right)\left(\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-T/\tau} z^{-1}}\right) = K\frac{\left(1 - e^{-T/\tau}\right) z^{-1}}{1 - e^{-T/\tau} z^{-1}}$$

3 Catalog of G(z)'s

$P(s)$	$G(z)=Z\{L^{-1}[H(s)P(s)]\}$
$\frac{K}{s}$	$KT\frac{z^{-1}}{1-z^{-1}}$
$\frac{K}{\tau s+1}e^{-kTs}$	$K\frac{(1-e^{-T/\tau})z^{-(k+1)}}{1-e^{-T/\tau}z^{-1}}$
$\frac{K}{(\tau_1 s+1)(\tau_2 s+1)}e^{-kTs}$	$K(1-z^{-1})z^{-k}\left(\frac{1}{1-z^{-1}}+\dots \frac{\tau_1}{\tau_2-\tau_1}\left(\frac{1}{1-e^{-T/\tau_1}z^{-1}}\right)-\frac{\tau_2}{\tau_2-\tau_1}\left(\frac{1}{1-e^{-T/\tau_2}z^{-1}}\right)\right)$

Note: The hold device does not effect stability. However, zeros may result from sampling.

3 Zeros Resulting From Sampling

Example 9.8 Let $p(s)=\frac{1}{(10s+1)(25s+1)}$. Using MATLAB, compute the effective discrete transfer function for two sample intervals: (a) T = 3 and (b) T = 10. Comment on the results.

Solution: Using the object-oriented LTI system.

```
» s=TF('s'); p=1/(10*s+1)/(25*s+1);
» sys=c2d(p,3,'zoh')
   0.01568 z + 0.01363
 ----------------------
 z^2 - 1.628 z + 0.657
 Sampling time: 3
```

$$p(z)=\frac{0.0157(z+0.869)}{(z-0.889)(z-0.739)}$$

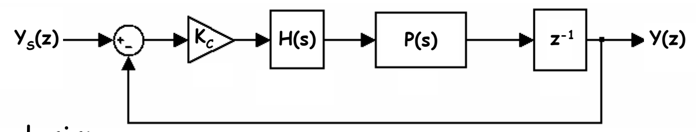
```
» sys=c2d(p,10,'zoh')
   0.1281 z + 0.08034
 ----------------------
 z^2 - 1.038 z + 0.2466
 Sampling time: 10
```

$$p(z)=\frac{0.128(z+0.627)}{(z-0.670)(z-0.368)}$$

4 Closed Loop Response

Example 9.9:

A first order process is controlled using a discrete controller, at a sample rate of T = 1 min. The effluent composition is measured using a GC, with a delay of one sample. The process characteristic time is 1.443 min.



Analysis:

$$G(z) = Z\left\{L^{-1}\left\{H(s)P(s)\right\}\right\} = K\frac{(1-b)z^{-1}}{1-bz^{-1}}, b = e^{-T/\tau}$$

$$\frac{Y(z)}{Y_S(z)} = \frac{K_C G z^{-1}}{1+K_C G z^{-1}} = \frac{K_C K(1-b)z^{-2}}{1-bz^{-1}+K_C K(1-b)z^{-2}}$$

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4 Closed Loop Response

$$\frac{Y(z)}{Y_S(z)} = \frac{K_C G z^{-1}}{1+K_C G z^{-1}} = \frac{K_C K(1-b)z^{-2}}{1-bz^{-1}+K_C K(1-b)z^{-2}}$$

For $K = 1, \tau = 1.443, T = 1$, so $b = 0.5$: $\frac{Y(z)}{Y_S(z)} = \frac{0.5K_C}{z^2 - 0.5z + 0.5K_C}$

The closed-loop characteristic equation has two roots:

$$p_{1,2} = 0.25 \pm \sqrt{0.25 - 2K_C}\,/2$$

For $K_C > 0.125$: $p_{1,2} = 0.25 \pm i\sqrt{2K_C - 0.25}\,/2$

For step changes in the set point, $Y_S(z) = z/(z-1)$,

Hence, the FVT gives: $\lim_{z\to 1} Y(z) = \frac{K_C}{1+K_C}$

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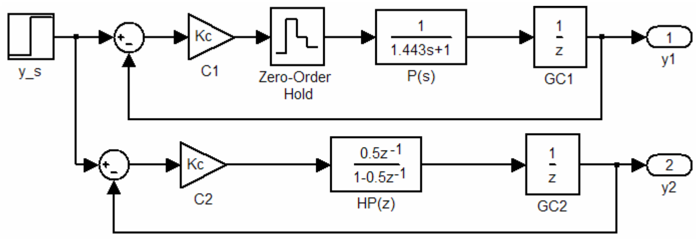
4 Closed Loop Response

SIMULINK Model.



Note that the two loops above are equivalent and produce the same results!

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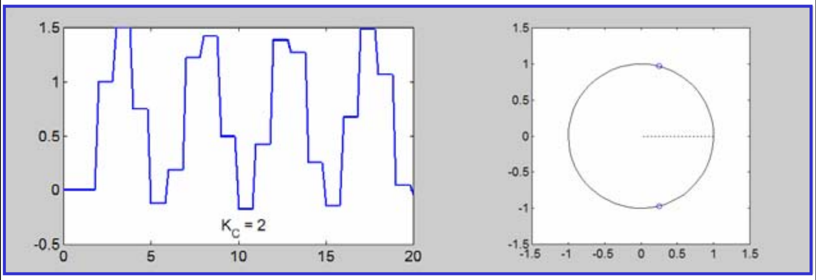
4 Closed Loop Response

Simulation results.



As the controller gain is increased, the offset is reduced at the expense of more oscillatory response. The closed-loop poles are displayed on the right, and show that at K_C = 2, the system is at the stability limit.

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Summary

On completing this section, you should:

❶ Be able to represent continuous and discrete signals in the same framework (Z-transforms).

❷ Have a working knowledge of Z-transforms and their usage in the analysis of discrete systems.

- The ability to perform manipulations using Z-transforms is a basic analytical skill required in this course!

❸ Be able to compute the transfer function of an arbitrary system involving continuous and discrete components.

- First compute discrete equivalents of continuous transfer functions, and then combine all discrete elements.

❹ Be able to compute the response of a discrete system (open and closed loop).

- Either by inverse Z-transforms or using SIMULINK.

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